PLANCKS 2018

Zagreb, Croatia

Instructions

In the envelope you can find three types of sheets:

- blank concept papers
- papers for official solutions
- enumerated papers from 1 to 10.

You can write anything you want on the blank concept papers (nobody cares about them). Your official solutions have to be written on the official papers which have to be put into the corresponding enumerated paper (e.g. solution of the problem number 4 goes into paper 4).
Also, **don't forget to put your team name on the envelope and on each paper** (official papers and enumerated papers).

GL HF <3

1 Harmonic Oscillator

prof. dr. sc. Hrvoje Buljan *

Consider a quantum particle of mass m and charge q trapped in a two-dimensional ($2D$) harmonic oscillator potential:

$$V(x,y) = \frac{1}{2} m\omega^2 \left(x^2 + y^2\right) \ .$$

An infinite solenoid is piercing the xy plane at $x = y = 0$. The solenoid is infinitelly thin; its magnetic field is $\mathbf{B} = \Phi\delta(x)\delta(y)\hat{z}$.

1. Write down the Schrödinger equation for the particle if the vector potential (in polar coordinates (r, ϕ)) is chosen to be $\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2r\pi}\hat{\phi}$.
   Consider a gauge transformation generated by the function $\Lambda(\mathbf{r})$: $\mathbf{A}'(\mathbf{r}) = \mathbf{A} + \nabla\Lambda$. What is the relation between the wavefunctions ψ and ψ', which are solutions of the Schrödinger equation in the gauge $\mathbf{A}$ and $\mathbf{A}'$, respectively? **(3 points)**

2. Consider a singular gauge transformation given by $\Lambda = -\frac{\Phi}{2\pi}\phi$. Write the Schrödinger equation for the particle in this gauge. Find the boundary condition for the particle's wavefunction when $\phi \to \phi + 2\pi$. Show that $z^\alpha f(|z|)$, where $z = x + iy$, satisfies this boundary condition for an appropriate value of α. By using this fact, construct an eigenstate for this system in this gauge. Can you argue that this eigenstate is the ground state when the flux Φ is sufficiently small. **(4 points)**

3. Write the wavefunction from the previous item in the gauge $\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2r\pi}\hat{\phi}$, and calculate the expectation value of the canonical and the physical angular momentum $\left\langle \hat{L}_z \right\rangle$. **(1 points)**

4. If the solenoid current would be suddenly turned off to zero at $t = 0$, what would be the appropriate wavefunction to describe the system at $t = 0^+$. Calculate the expectation value of the canonical and the physical angular momentum ($\left\langle \hat{L}_z \right\rangle$) at $t = 0^+$ and explain the result. What would be the final state of the system if the flux is adiabatically turned off to zero? **(2 points)**

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2 *Trans*-polyacetylene

prof. dr. sc. Ivo Batistić *

Trans-polyacetylene is a polymer, a large molecule built of repeated units made of carbon and hydrogen units (CH), as it is shown in Figure (1):

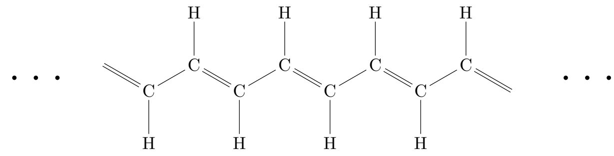

Figure 1: Basic structure of *trans*-polyacetylene.

In the ground state, the polyacetylene is a dimerized chain with alternating short and long bonds between carbon atoms. The dimerization is a result of the Peierls instability†. Each carbon atom has p_z‡ atomic orbital that is occupied by a **single electron**. There is an overlap between p_z orbitals from the neighboring carbon atoms, which enables electrons to jump from one atom to another, and to propagate along the chain. The electron states to be addressed in this problem are composed of these p_z orbitals. All other orbitals are fully occupied or empty and they can be neglected.

The atomic orbitals are not stationary states. The atomic orbitals make sense only for isolated atoms. For molecules, crystals or polymers the wave function of an electron can be approximated as a linear combination of atomic orbitals. Each atomic orbital enters the linear combination with a coefficient/an amplitude ψ_n (n is the atomic index). In the tight-binding approximation (TBA), unknown amplitudes ψ_n, satisfy the following equations (for one-dimensional chain):

$$e \cdot \psi_n = -t_{n,n-1}\,\psi_{n-1} - t_{n,n+1}\,\psi_{n+1} \qquad n = 0, \pm 1, \ldots (n \text{ is index of the atom})$$

TBA equations represent the eigenvalue problem for the electron energies e. $|\psi_n|^2$ is the probability of finding the electron at n-th atom. $t_{n,n\pm 1}$ is an amplitude for electron hopping from n-th atom to $(n \pm 1)$-atom. Since the overlap between atomic orbitals depends on the distance between atoms, the hopping amplidutes, $t_{n,n\pm 1}$, are also distance dependent.

1. Apply tight-binding approximation to polyacetylene for the electrons in p_z orbitals, and write down the corresponding TBA equations. **(1 point)**
   Apply the Bloch theorem to the wave functions amplitudes, and write down the eigenvalue problem for the periodic part of the Bloch wave function. **(1 point)**
   Find out how the electron energy, e, depends on the wave number. **(1 point)**
   What is the energy gap between occupied and unoccupied states if hopping amplitudes for short and long bonds are 2.875 eV and 2.125 eV respectively. **(1 point)**

   Hint: Polyacetylene is a dimerized chain with periodicity of two (CH)-units. The wave number should be defined with respect to the unit cell with two carbon atoms. Assume that there are two dinstinct hopping amplitudes for short and long bond, for example, t_1 and t_2.

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†Peierls' theorem states that a one-dimensional equally spaced chain is unstable with respect to the periodic lattice deformation that opens a gap at the Fermi level.

‡p_z orbital extends in the direction perpendicular to the plane defined by the zig-zag bonds connecting carbon atoms and bonds connecting carbon and hydrogen atoms.

2. What is the average electron energy of **undimerized** polyacetylene. **(1 point)**
   Assume that hopping amplitude for undimerized chain is 2.5 eV.
   What is the average electron energy of dimerized polyacetylene. **(1 point)**

   Hint: Elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} dx\,\sqrt{1 - k^2 \sin^2 x}$$

   The asymptotic expansion for k close but less than 1 ($k' = \sqrt{1-k^2}$):

$$E(k) \approx 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'} - \frac{1}{2}\right)$$

3. The polyacetylene is a topological insulator*. A topologically different state is obtained when the long bond becomes short, and the short bond becomes long. The both topological states are shown in Figure (2):

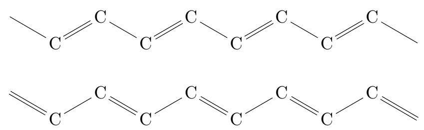

Figure 2: Topologically different states of polyacetylene. For the sake of simplicity hydrogen atoms are omitted.

   The Hamiltonian for the periodic part of the Bloch wave function can be written as:

$$H(q) = \sigma_x\, h_x(q) + \sigma_y\, h_y(q)$$

   Write down functions $h_x(q)$ and $h_y(q)$. **(1 point)**

   What is the winding number † around the origin, (0,0), of the curve ($h_x(q)$,$h_y(q)$) for each topological state, when q is running over all wave vectors in the first Brillouin zone, from negative to positive values. **(1 point)**

   Hint: Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$$

4. Consider polyacetylene chain with a defect separating two topological states, as it is shown in Figure (3):

*Two insulators are in the same topological class if they can be mapped to each other by a continuous change of Hamiltonian parameters keeping the energy gap finite.

†The **winding numer** of a closed curve around a **given point** is an integer representing the total number of times that curve travels counterclockwise around the point.

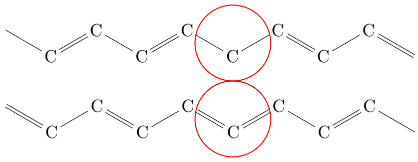

Figure 3: Two types of topological defects (labeled with circles) in polyacetylene separating the different topological states.

Demonstrate the existence of the electron state with the energy $e = 0$ (within the gap; also called **edge state**), localized around the defect for both defect types.

(2 points)

3 The Unruh effect

Grgur Šimunić, mag. phys. [*]

Consider a $(1+1)$ dimensional flat spacetime with coordinates (t, x) and metric (we will use the term metric for the metric tensor, not the actual metric):

$$g_{ab} = -(\mathrm{d}t)_a(\mathrm{d}t)_b + (\mathrm{d}x)_a(\mathrm{d}x)_b\,. \tag{1}$$

Here (and in what follows) we use the abstract index notation to denote the tensor types and $(\mathrm{d}t)_a$ and $(\mathrm{d}x)_a$ are coordinate basis one-forms on cotangent bundle over spacetime manifold and the product of 1-forms is understood to be a tensor product.[†] We are also using natural unit system ($c = G = \hbar = k_B = \epsilon_0 = 1$). Next, consider an observer moving with constant proper acceleration a^a in this spacetime and let (τ, ξ) be the coordinates meassured by this observer. Furthermore, let $a = \sqrt{|a^c a_c|}$.

The purpose of this problem is to determine how this accelerated observer sees certain physical effects. To be more specific, if an inertial observer sees some physical system in a ground state, we will determine in what state will the accelerated observer see the same system. The physical system of interest here will be the system made of elementary particles which we describe by certain fields. Here we will only consider the simplest case of massless scalar particles.

1. Find the transition functions between the laboratory frame and the frame of the accelerated observer. Express the metric in the (τ, ξ) coordinates. Assume the following initial conditions for the accelerated observer:

$$x^\mu(\tau = 0) = \left(0, \frac{1}{a}\right)\,, \tag{2}$$

$$\left.\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right|_{\tau=0} = (1, 0)\,. \tag{3}$$

(1 point)

2. Find the function $\bar{\xi}(\xi)$ such that the metric g_{ab} is proportional to $(-(\mathrm{d}\tau)_a(\mathrm{d}\tau)_b + (\mathrm{d}\bar{\xi})_a(\mathrm{d}\bar{\xi})_b)$ and determine the proportionality factor (assume $\bar{\xi}(\xi = 0) = 0$). What are the transition functions between (t, x) and $(\tau, \bar{\xi})$ frames? **(1 point)**

3. Consider a massless, real scalar field ϕ described by the action:

$$S[\phi] = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a\phi(x)\partial_b\phi(x)\,, \tag{4}$$

where g denotes the determinant of the metric. Show that ϕ can be expressed in following forms:

$$\phi(t, x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\sqrt{|k|}}\left(a_k^- \mathrm{e}^{ikx - i|k|t} + a_k^+ \mathrm{e}^{-ikx + i|k|t}\right)\,, \tag{5}$$

$$\phi(\tau, \bar{\xi}) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{\sqrt{|k|}}\left(b_k^- \mathrm{e}^{ik\bar{\xi} - i|k|\tau} + b_k^+ \mathrm{e}^{-ik\bar{\xi} + i|k|\tau}\right)\,, \tag{6}$$

where $a_k^-, a_k^+, b_k^-, b_k^+ \in \mathbb{C}$. Are a_k^- and a_k^+ independent? If so, explain why, and if not, determine the relation between them. **(2 points)**

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[†]Not infrequent, this equation is written in the form $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$. However, this should not be written like this. Even though the square of the forms $\mathrm{d}t$ and $\mathrm{d}x$ is well defined as $(\mathrm{d}t)_a(\mathrm{d}t)^a$ and $(\mathrm{d}x)_a(\mathrm{d}x)^a$, respectively, the square of $\mathrm{d}s$ is problematic because the form with such square does not necessarily exists. And even though there is a way around that problem locally, one should avoid such notation.

4. Let us preform a quantization of this field by making $\phi, a_k^{\pm}, b_k^{\pm}$ operators. Assume canonical commutation relations:

$$[\phi(t,x), \phi(t,y)] = [\dot{\phi}(t,x), \dot{\phi}(t,y)] = 0\,, \tag{7}$$
$$[\phi(t,x), \dot{\phi}(t,y)] = i\delta(x-y)\,, \tag{8}$$

where dot denotes derivation with respect to t. How do commutation relations look like in $(\tau, \bar{\xi})$ coordinates? What are commutation relations for $a_k^{\pm}$ and $b_k^{\pm}$? Which of these can be identified as creation operators, and which as annihilation operators? **(1 point)**

5. Express the field ϕ as a function of lightcone coordinates:

$$u = t - x\,,\ \ v = t + x\,, \tag{9}$$
$$\bar{u} = \tau - \bar{\xi}\,,\ \ \bar{v} = \tau + \bar{\xi}\,. \tag{10}$$

(1 point)

6. Show that the operator b_Ω^- can be expressed as:

$$b_\Omega^- = \int_0^\infty \mathrm{d}\omega \sqrt{\frac{\Omega}{\omega}} \left(a_\omega^- F(\omega,\Omega) + a_\omega^+ F(-\omega,\Omega) \right)\,, \tag{11}$$

where F is some complex function of two variables. Determine the function F. **(1 point)**

7. Define the vacua states for laboratory and accelerating observers. If the field is in the vacuum state of the laboratory observer, determine what is the particle density n_Ω of particles with momentum Ω that the accelerated observer measures. How can we interpret these results?

   Hint: In order to compute n_Ω, first show that $F(\omega,\Omega)$ and $F(-\omega,\Omega)$ are proportional and determine the proportionality factor. **(3 points)**

4 Redshifted 21-cm signal from the Epoch of Reionization

dr. sc. Vibor Jelić *

A number of radio telescopes (e.g. LOFAR in the Netherlands; MWA in Australia; and HERA in South Africa) are aiming to detect the redshifted 21-cm hyperfine line of neutral hydrogen from the Epoch of Reionization (EoR), a pivotal period in the history of the Universe during which all-pervasive cosmic gas was ionized by radiation of the first “stars”.

1. Calculate the range of frequencies to which a radio telescope needs to be sensitive to detect the cosmological 21-cm signal. Current observational constrains suggest that the EoR occurred roughly within the redshifts of 6 and 15. **(2 points)**

2. Write down the equation of radiative transfer along a line of sight through a hydrogen cloud of *optical depth* τ_ν, defined as the integral of the absorption coefficient (α_ν) along the path through the cloud. The *brightness* (or *specific intensity*) I_ν of emission emerging from the cloud at frequency ν should be quantified by the equivalent *brightness temperature* $T_b(\nu)$ such that $I_\nu = B_\nu(T_b)$, where B_ν is the Planck function. For the background emission consider only the cosmic microwave background (CMB), emission of the universe as a black body. Assume the absence of any scattering along the path. For an illustration see Figure 1. **(2 points)**

3. Solve the equation of radiative transfer, yielding the brightness temperature of the emergent radiation at frequency ν. Assume uniform excitation temperature $T_{\rm ex}$ through a cloud. The excitation temperature of the 21 cm line is known as the spin temperature T_S. The spin temperature is defined through the ratio between the number densities n_i of hydrogen atoms in the two hyperfine levels (1S singlet, n_0, and 1S triplet levels, n_1). Write down the equation for the spin temperature, if the ratio of the statistical degeneracy factors of the two levels is $g_1/g_0 = 3$. **(3 points)**

4. What happens if the intervening cloud and the cosmic microwave background radiation are in thermodynamic equilibrium? Discuss if the measurement in such a case reveals anything interesting about the intervening cloud? If not, propose conditions in which the measurement will be successful. Explain why. **(3 points)**

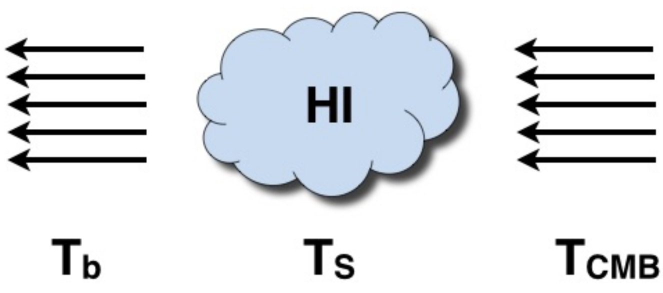

Figure 4: An illustration of various components relevant for the radiative transfer problem: the background radiation (CMB) going through a hydrogen cloud of optical depth τ_ν and emerging with the brightness temperature T_b. The excitation temperature of the 21 cm line associated with the hydrogen cloud is known as the spin temperature T_S.

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5 Motion of point charge in the presence of the magnetic monopole

dr. sc. Bruno Klajn *

During the undergraduate course in physics, the student is familiarized with physical systems of fundamental importance, such as the Kepler problem and the harmonic oscillator, both of which are exactly solvable. There is, however, an additional system on the same level of importance which is rarely mentioned in undergraduate textbooks. It is the motion of point charge q in the presence of a fixed magnetic monopole g. In this problem, you will investigate some elementary properties of this system.

Let the magnetic monopole g be fixed at the origin of coordinates so that it produces the magnetotstatic field

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^3}\vec{r},$$

with $\vec{r}$ being the position vector of arbitrary point in space. A point particle of electric charge q and m is free to move in the presence of this field. Assume that all motion is nonrelativistic.

1. Demonstrate that the Lorentz force acting on the particle is perpendicular to its velocity, $\vec{F} \perp \vec{v}$, and then, using this fact, show that the kinetic energy T of the particle is a constant of motion. Furthermore, show that this implies that the particle always moves at constant speed, equal to its initial speed $|\vec{v}| = v$. **(1 point)**

2. Similarly, show that the Lorentz force is perpendicular to the instantaneous position of the particle, $\vec{F} \perp \vec{r}$. Using this, and writing the position vector of the particle in terms of its magnitude and direction, $\vec{r} = r\hat{r}$, derive the differential equation for $r(t)$. Solve this equation with the initial conditions at $t = 0$, $r(0) = r_{\min}$ and $\dot{r}(0) = 0$. From the solution $r(t)$, determine the physical meaning of $r_{\min}$. **(2 points)**

3. Show that while the torque $\vec{\tau}$ acting on the particle is nonvanishing, it can nevertheless be written as a derivative of a certain vector, $\vec{\tau} = -\mathrm{d}\vec{L}_{\mathrm{em}}/\mathrm{d}t$, which implies that the conserved quantity for this motion is not the particle angular momentum $\vec{L}$ itself, but rather the combination $\vec{J} = \vec{L} + \vec{L}_{\mathrm{em}}$. Explicitly calculate the vector $\vec{L}_{\mathrm{em}}$. Verify that the vector $\vec{L}_{\mathrm{em}}$, is, in fact, the angular momentum of the electromagnetic field defined by the integral

$$\epsilon_0 \int_{\mathbb{R}^3} \vec{r}' \times \left[\vec{E}(\vec{r}') \times \vec{B}(\vec{r}')\right] \mathrm{d}V',$$

where $\vec{E}$ is, in the nonrelativistic approximation, just the electrostatic field of charge q. Conclude that the vector $\vec{J}$ represents the total angular momentum of the system and calculate its magnitude J in terms of known quantities and initial conditions. **(3 points)**

4. Now introduce the coordinate axes with the z-axis pointing along the vector $\vec{J}$ and use the standard spherical coordinates in what follows. Calculate the quantity $\hat{r} \cdot \vec{J}$ and use it to find the time dependence of the zenith angle $\theta(t)$. Is the motion of the particle constrained to some surface? If so, identify the surface in question. **(2 points)**

5. Finally, from the expression $\hat{r} \times \vec{J}$ determine the equation of motion for $\hat{r}$. Rewrite the obtained equation in the coordinate system introduced above and obtain the

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differential equation for the azimuthal angle $\varphi(t)$. Solve the differential equation with the initial condition $\varphi(0) = 0$. **(2 points)**

After following these steps, you have found all constants of motion and calculated the trajectory of the particle. Therefore, you have completely solved the problem of charged particle motion in the presence of magnetic monopoles.

6 Cosmic Magellan

prof. dr. sc. Krešimir Kumerički *

Imagine that propagation of light through our homogenous and isotropic universe is described by the differential relativistic interval:

$$ds^2 = c^2 dt^2 - a(t)^2 \left[dr^2 + R_0^2 \sin^2\left(\frac{r}{R_0}\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] = 0\,,$$

where r is the radial coordinate (defined as *comoving*, which means that r of a given object doesn't change over time), and $a(t)$ is the scale factor, which encodes expansion of the universe with time t as

$$a(t) = \frac{c}{R_0} t\,,$$

with c being the speed of light and R_0 constant length parameter. We assume that the universe is closed and finite, with maximal value of r equal to $R_0\pi$.

1. In this case we would in principle be able to see the whole picture of our galaxy somewhere on the night sky, thanks to its light travelling all the way around the universe. What would be the relative shift (redshift) $z \equiv \frac{\Delta\lambda}{\lambda}$ of wavelengths of this light?
   Hint: $1 + z = a(t_0)/a(t_e)$, where t_e and t_0 denote times of emission and observation of light signals. **(5 points)**

2. Other galaxies we would be able to see at least twice: once via light taking the shortest path, with redshift z_1, and again, via light taking one turn around the universe, with redshift z_2. Show that these two redshifts are related to the one from the (a) part as $(1 + z_1)(1 + z_2) = (1 + z)$. **(5 points)**

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7 Solar neturinos

prof. dr. sc. Matko Milin *

In cores of the Sun and similar stars, energy is mainly released through the so-called "ppI-cycle", *i.e.* the following series of nuclear reactions:

$$\mathrm{p} + \mathrm{p} \to \mathrm{d} + \mathrm{e}^+ + \nu \quad ,$$
$$\mathrm{p} + \mathrm{d} \to {}^3\mathrm{He} + \gamma \quad ,$$
$${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + 2\mathrm{p} \quad .$$

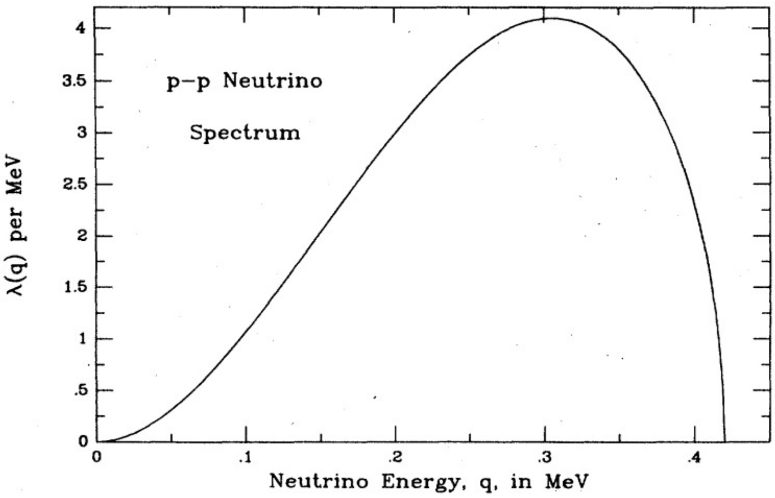

Figure 5: Neutrinos produced in the ppI-cycle have the energy spectrum given in figure (taken from Bahcall and Ulrich, Rev. Mod. Phys. 60, 297, 1988.) - the integral of $\lambda(q)\mathrm{d}q$ is here normalized to unity for $\mathrm{d}q$ measured in MeV.

The theory of beta-processes describes the curve in the figure approximately by the formula:

$$\lambda = 193.985\,\mathrm{MeV}^{-5} \cdot q^2(Q + m_e c^2 - q)\sqrt{(Q + m_e c^2 - q)^2 - (m_e c^2)^2} \qquad ,$$

where Q is the Q-value of the first step in the cycle ($Q\approx$ 420 keV), and $m_e c^2\approx$ 511 keV is the energy corresponding to the mass of the electron.

1. Find the average energy of the neutrinos produced in the cycle! **(4 points)**
   If you can't calculate it, estimate it from the figure (as you'll need it later).

2. Find the (average) energy that is added to the solar core material (plasma) for every produced ^{4}He nucleus. The masses of the particles involved in the ppI-cycle are: $m(\mathrm{p})= 1.6726\cdot 10^{-27}$kg, $m(\mathrm{d})= 3.3445\cdot 10^{-27}$kg, $m(^3\mathrm{He})= 5.0082\cdot 10^{-27}$kg, $m(^4\mathrm{He})=$ $6.6465\cdot 10^{-27}$kg, $m(\mathrm{e}^+)= 9.1094\cdot\ 10^{-31}$kg. **(2 points)**

3. The so-called "solar constant" is the mean flux of solar electromagnetic radiation per unit area, measured on a surface perpendicular to the rays, one astronomical unit from the Sun (roughly the distance from the Sun to the Earth, d= $150\cdot 10^6$km). It is easily measured to be $\approx$ 1360 W/m^2. How much mass is converted into energy every second in the Sun, assuming that all the solar energy is coming from the ppI-cycle? **(0.5 points)**
   How many ppI-cycles happen every second? **(0.5 points)**

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4. Assuming that the Sun will convert 10% of its initial hydrogen into helium during its evolution, estimate the lifetime of the Sun. The solar mass is $2{\cdot}10^{30}$kg and the initial mass of the hydrogen was ≈75% of that number. **(1 point)**

5. How many neutrinos does the Sun emit every second? **(0.5 points)**
   What is the solar neutrino density flux (*i.e.* the number of neutrinos per second and per unit area perpendicular to their velocity) at Earth? **(0.5 points)**
   The obtained number corresponds to the upper limit of the flux of solar electron neutrinos (their number gets smaller due to the neutrino oscillations).

6. What is the energy flux (energy per unit area per unit time) associated with the solar neutrino density flux at Earth? **(0.5 points)**
   What share of total emitted energy comes from the Sun through neutrinos? **(0.5 points)**

8 Motion on a torus

prof. dr. sc. Tamara Nikšić *

An ordinary ring torus embedded in three-dimensional space can be described by the parametric equations

$$
\begin{aligned}
x &= (c + a\cos v)\cos u, \\
y &= (c + a\cos v)\sin u, \\
z &= a\sin v,
\end{aligned}
$$

for $u, v \in [0, 2\pi\rangle$ and $c > a$ (both c and a are positive constants). Some particle of mass m is constrained to move on the surface of such a torus and is not subjected to external forces (e.g. gravity).

1. Write down the Lagrangian and equations of motion. **(1 point)**
2. Derive the constants of motion and reduce the equation of motion to the equivalent one-dimensional problem. **(1 point)**
3. Use the effective potential to discuss the qualitative nature of the orbits. Consider all possible cases. **(3 points)**
4. Assume that the energy of the particle equals $E = \dfrac{p_u^2}{2m(c-a)}$ (p_u denotes the generalized momentum) and that the particle starts from the $v = 0$ point (u is arbitrary). Calculate the time needed for the particle to reach the $v = \pi$ position. **(3 points)**
5. Finally, assume that the homogeneous gravitational field is switched on and discuss qualitatively its influence on the particular orbits. **(2 points)**

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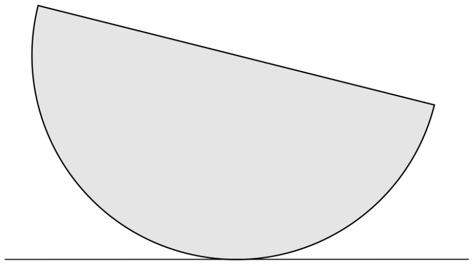
9 Almost oscillating

doc. dr. sc. Nikola Poljak *

A solid (three-dimensional) semi-ball is placed on ice (in the homegeneous gravitational field $\vec{g}$), so that there is absolutely no friction between the object and the surface as in the figure.



1. Determine the position of the center of mass of a semi-ball of radius of curvature r. **(1.5 points)**
2. Determine the moment of inertia of the semi-ball around an axis passing through the center of mass which is perpendicular to the axis of symmetry of the semi-ball. The mass of the semi-ball is m. **(2.5 points)**
3. Write down the total energy of the semi-ball while it is left to move on its own and describe the movement in words. **(2 points)**
4. The semi-ball will perform a movement that is almost oscillating. Write down the equation of motion of the semi-ball. **(1.5 points)**
5. The solution to this equation is very complicated, however, the semi-ball is almost oscillating. First, assume that the oscillations are small. Next, assume that the solution is given by $\theta(t) = \theta_0 \cos(\omega t)$. Plug this solution into the equation of motion and obtain the frequency of oscillation. θ is the angle between the axis of symmetry of the semi-ball and $\vec{g}$. **(2.5 points)**

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10 Phase transitions in contagion models using complex networks

Matija Medvidović, mag. phys.
prof. dr. sc. Davor Horvatić *

In this problem, we are going to calculate the time evolution of a complex network describing, for example, propagation of a disease through a population. To start things off, let us define a complex network as a graph for use in this problem. Graphs consist of:

- **Vertices** - labeled points in a plane, denoted by, for example, (i)
- **Edges** - labeled lines connecting two vertices, denoted by, for example, (i, j)

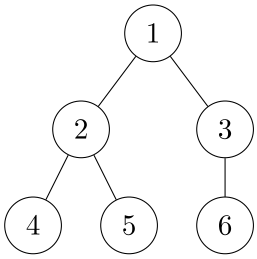
Graphs can be viewed as generalized and irregular crystal lattices. Their geometrical structure is captured by the **adjacency matrix** defined as:

$$A_{ij} = \begin{cases} 1 & \text{, if vertices } (i) \text{ and } (j) \text{ are connected by an edge} \\ 0 & \text{, otherwise} \end{cases} \tag{12}$$

Additionally, we assign a quantity called the **degree** k_i to each vertex (i):

$$k_i = \sum_j A_{ij} \tag{13}$$

Simply put, k_i is the number of edges attached to the vertex (i).



$$\Longrightarrow \quad A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to k_1 = 2 \\ \to k_2 = 3 \\ \to k_3 = 2 \\ \to k_4 = 1 \\ \to k_5 = 1 \\ \to k_6 = 1 \end{matrix}$$

Statistical properties of networks are commonly described by the **degree distribution** p_k - it is the probability that a randomly selected vertex from a given network will be of degree k.

Consider a network with a general degree distribution p_k. Let each vertex (i) have an additional internal variable $\sigma_i = 0, 1$ denoting its "activity" ($\sigma_i = 0$ means the vertex is inactive and $\sigma_i = 1$ means it is active).

Interpret the vertices as individual persons and edges as "coming into contact" (geographical proximity, common workplace etc.). If the person (i) is not affected by a disease (active) and it comes into contact with a sufficient number of sick people (inactive) then it may become sick itself.

Let us define the following processes of transferring the disease:

- A healthy person can become sick **internally** during the time interval Δt with a constant probability $p\Delta t$.

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- A healthy person can become sick **externally** during the time interval Δt with probability $r\Delta t$ if she/he has less or equal to m healthy neighbors. We assume that the network is dense enough for $m < \min(k_i)$ to hold (social networks usually are).
- A sick person becomes healthy during the time interval Δt with probability $q\Delta t$.

So, to summarize, we define two ways of getting sick, one certain way of becoming healthy again by introducing four parameters: probabilities p, r, q and an integer m defining the number of sick people one has to come in contact with in order to become sick.

Answer the following questions with the above described network representation of the problem:

1. Denote the number of sick people at time t by $N_S(t)$ in a population (network) of N people and the fraction of sick people at time t by $x(t) = {}^{N_S(t)}/_{N}$. You can approximate the probability that a randomly chosen vertex is sick by x.

   Assuming that the sick and healthy people are well mixed in the network, write down the approximate probability of a randomly chosen vertex of degree k (connected to k other people/vertices) to have **exactly** n healthy vertices in its neighborhood in terms of k, n, and x. Does the resulting distribution of n have a name?

   Taking that into account, what is the probability $\varphi_m^{(k)}(x)$ of there being less or equal to m sick people in the neighborhood of that vertex? **(1 point)**

2. Using that result, write down the expected number of sick people N_S at time $t + \Delta t$ using N_S at the previous timestep t. At each timestep, use

$$\varphi_m(x) \equiv \langle \varphi_m^{(k)}(x) \rangle_k = \sum_k p_k \, \varphi_m^{(k)}(x) \tag{14}$$

   as the probability a random vertex will have at less or equal to m healthy neighbors. **(2 points)**

3. Once you have arrived at the discrete-time representation of the equation, show that by taking the limit $\Delta t \to 0$ one obtains the following dynamical equation for x:

$$\frac{dx}{dt} = r(1-x)\varphi_m(x) + p(1-x) - qx \tag{15}$$

   (1 point)

4. For simplicity, from now on assume that $q = 1$ and that the underlying network is a **regular graph**: each vertex is of degree k_0, exactly (it has k_0 neighbors). This is equivalent to the cubic latice with dimensionality $d = {}^{k_0}/_{2}$ in condensed matter physics. What is the degree distribution p_k of such a graph? Does this choice simplify the expression for $\varphi_m(x)$?

   - For $m = k_0$, find the fixed points $x_s(p, r)$ in parameter space defined by $\dot{x} = 0$.

- Write down the equation for fixed points in the case of $m = k_0 - 1$. Does a nonzero solution always exist? Find the critical value $r_c = r_c(k_0)$ at which the equation gets a nonzero solution for $p = 0$?

Hint: Try representing the equations graphically and look for curve intersections.

(3 points)

5. Finally, in the case of $m = k_0 - 1$, define the critical exponents β and δ by approximating $x_s(r, p)$ near the critical point ($r = r_c$ and $p = 0$, respectively):

$$x_s(r, p = 0) \propto |r - r_c|^{\beta} \tag{16}$$

$$x_s(r = r_c, p) \propto p^{1/\delta} \tag{17}$$

Hint: It is worth noticing that the value of x_s is small near the critical point $r = r_c$.

What are the values of β and δ? What is the order of such a phase transition? Briefly comment on the difference/similarity between these values and the well-known universal mean-field exponents often encountered in statistical physics.

(3 points)